Acoustic Wave Amplification in Solid Rockets by Discrete Mass Injection

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Discrete mass injection as a contributing source of transverse acoustic wave excitation in a solid-propellant rocket motor has been studied for a particular class of grain configurations. This configuration consists of a circular cylindrical chamber at the foreward end of which a series of radial slots are cut into the propellant. During burning, combustion gases generated within the slots are injected into the cylindrical chamber as plane jets. Since the slot burning area may represent a significant fraction of the total propellant burning surface, flow velocities into the chamber may be large, especially early in the motor run. The analysis shows that the mass flux from the slots convects energy that contributes to the driving of acoustic waves in the cylindrical chamber. In addition, depending on the number of radial slots present, certain standing transverse modes are driven at higher rates than others. For traveling transverse wave modes, no selective driving effects can be found. In general, traveling waves are more prone to amplification by discrete mass injection than standing waves. Longitudinal modes may also be excited by the same mechanism. Clearly the influence of grain geometry on acoustic instability should be included among other factors in the design of slotted grain configurations.

Nomenclature

A	=	admittance function
A_α	=	coefficients of Green's function
a_0	=	speed of sound
E_α	=	normalization constant defined by Eq. (21)
g	=	function defined by Eq. (11)
h	=	function defined by Eq. (13)
K	=	complex frequency
k	=	eigenvalues
L	=	chamber length
l,m,n	=	wave mode numbers
N	=	number of slots
P	=	pressure
$p^{(1)}$	=	pressure defined by Eq. (8)
$\bar{q}^{(1)}$	=	velocity perturbation defined by Eq. (8)
r	=	radial coordinate
R	=	chamber radius
t	=	time
$\bar{U}$	=	mean flow velocity
$\bar{u}$	=	velocity vector
z	=	longitudinal coordinate
γ	=	ratio of specific heats
ρ	=	density
η	=	index in Fourier Series
θ	=	azimuthal coordinate
ϕ	=	Fourier series, Eq. (4)
ϵ	=	small parameter proportional to wave amplitude
ν	=	small parameter proportional to mean flow Mach number
Λ	=	wave growth rate
Ψ	=	eigenfunction defined by Eq. (17)
Ω	=	wave frequency

Subscripts

0	=	reference conditions in absence of disturbance
s	=	refers to slot
w	=	refers to chamber wall

Superscripts

(r)	=	real part
(1)	=	order of term in powers of (ϵ)

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Introduction

THE propellant grain and combustion chamber of a solid-propellant rocket motor can assume a myriad of configurations. Considerations such as burning rate and flow dynamics of the gas stream will affect the grain design as a total system. Often times, an otherwise satisfactory configuration may cause undesirable amplification of acoustic waves naturally occurring within the grain cavity. As a result, certain wave modes may be driven to large amplitudes and induce undesirable vibration of the rocket motor.

A particular grain configuration which is of current interest consists of a circular cylindrical chamber at the foreward end of which a series of radial slots are cut into the propellant. A typical combustion chamber configuration is shown in Fig. 1. During operation of this type of propellant grain, combustion occurs within the slots as well as on the cylindrical chamber wall; so on the foreward end of the cylinder, mass is injected into the chamber from the slots in the form of upstream combustion gases. These plane jets issuing from the slots have relatively uniform velocity that may be assumed to have a steady mean value. The slot velocities may be quite large during the initial instants of motor operation, since the slot burning area is often a significant fraction of the total burning surface area.

Experimental data have indicated that during burning, certain specific transverse wave modes are driven to high amplitudes. Several studies (Refs. 1–5) have shown theoretically that burning at the cylindrical chamber wall contributes significantly to the amplification of transverse wave modes, but not all observed unstable modes can be accounted for by this driving mechanism alone. This paper will show that another source of energy in the amplification of transverse waves can be linked to the discrete mass addition into the chamber flow by the slot jets.

At first glance it may not be obvious that the slot jets can drive acoustic waves, but a simple physical mechanism can be envisaged if one recognizes that waves are amplified if there exists an oscillating mass flux from the chamber boundary with a component which is in phase with the pressure disturbance. This oscillating mass flux can arise from two sources: one representing the sensitivity of mass addition (propellant burning rate) at the burning surface within the slots to pressure fluctuations in the main chamber, and a second which results from interaction of the steady component of flow from the slot with the density fluctuations in the chamber. The latter effect is the principal concern in this paper. Considerable work remains in assessing the former part, since it involves a complex coupling of two or more acoustic cavities. However, the effects can be represented analytically by an admittance function written for the slotted chamber surface; a precise knowledge of the slot admittance function is not required in qualitatively evaluating the effect of the slot flow on transverse acoustic wave stability. Knowledge of the admittance is necessary only when accurate quantitative results are desired.

Following the usual practice of analyzing acoustic waves in a solid rocket combustion chamber, the combustion process is assumed to occur entirely within a thin boundary layer on the propellant surface. The mean chamber flowfield is assumed to be inviscid and incompressible.

The burning rate of propellant is pressure dependent, and the fluctuation of burning rate is coupled to the pressure oscillation through a burning surface admittance function. The wall motion due to mean burning rate of the propellant is negligible as compared to the velocity characteristic of the pressure oscillations, so the rate of recession of the burning surfaces will be ignored. In order to isolate the effects of discrete mass addition by the slot jets on the stability of transverse wave modes within the combustion chamber, we shall replace the combusting chamber wall by an inert surface. To be sure, the side wall is likely to be the primary source of driving energy as shown by many investigators,[2] but the effects which amplify or attenuate the acoustic waves are linear and additive to the order of precision of the calculations so that individual contributions to instability can be evaluated separately.

Analysis

The analysis will follow closely the small perturbation technique of Ref. 1. The coordinate system used is shown in Fig. 1. The equations of motion for an isentropic, compressible, and inviscid flowfield are a) continuity;

$$(\partial P/\partial t) + \gamma P \nabla \cdot \bar{u} = -\bar{u} \cdot \nabla P \tag{1}$$

and b) momentum;

$$\rho(D\bar{u}/Dt) + \nabla P/\gamma = 0 \tag{2}$$

Equations (1) and (2) are written in terms of dimensionless variables:

$$P = P'/P_0, \quad \bar{r} = \bar{r}'/R, \quad \rho = \rho'/\rho_0,$$
$$t = (a_0/R)t', \quad \bar{u} = \bar{u}'/a_0$$

where primes and subscript 0 denote dimensional and mean chamber values, respectively. The boundary condition can be specified by the velocities normal to the chamber surfaces, thus assuming all surfaces are inert except at the forward slot jets:

$$\hat{n} \cdot \bar{u} = \begin{cases} \nu \phi A P/\gamma & \text{at } z = 0 \\ 0 & \text{on all other surfaces} \end{cases} \tag{3}$$

where ν is a small parameter proportional to the Mach number of slot mass injection rate and A is the complex admittance function.

The function ϕ is a Fourier series describing the slot geometry given by

$$\phi = \frac{N\theta_s}{2\pi} + \frac{1}{\pi}\sum_{\eta=1}^{\infty} \frac{2}{\eta} \sin\left(\frac{\eta N \theta_s}{2}\right) \cos(\eta N \theta) \tag{4}$$

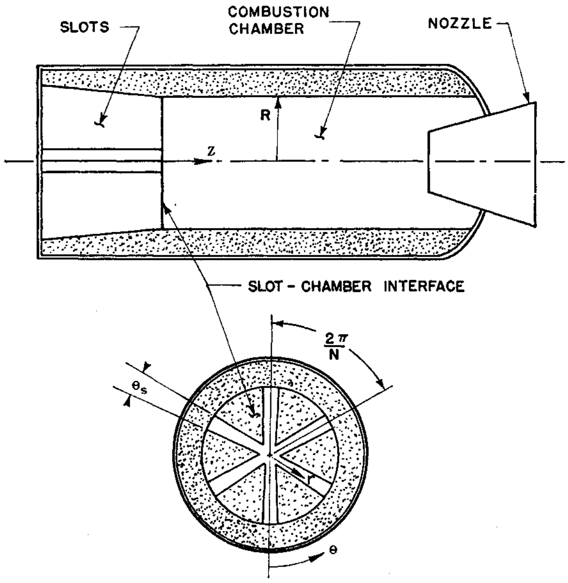

Fig. 1 Combustion chamber geometry.

where θ_s is the dimensionless slot width and N is the number of slots.

We next apply the linear analysis by writing the dependent variables P, ρ, and $\bar{u}$ as a combination of a mean part and a fluctuating part: $P = P_0 + \epsilon P^{(1)}$, $\bar{u} = \nu \bar{U} + \epsilon \bar{u}^{(1)}$, and $\rho = \rho_0 + \epsilon \rho^{(1)}$. Then Eqs. (1) and (2) become

$$[\partial P^{(1)}/\partial t] + \gamma \nabla \cdot \bar{u}^{(1)} = -\nu \bar{U} \cdot \nabla P^{(1)} \tag{5}$$

$$[\partial \bar{u}^{(1)}/\partial t] + \nabla P^{(1)}/\gamma = -\nu[\bar{U} \cdot \nabla \bar{u}^{(1)} + \bar{u}^{(1)} \cdot \nabla \bar{U}] \tag{6}$$

The relation between ν and ϵ is assumed to be

$$\lim_{\nu,\epsilon \to 0} \epsilon/\nu = 0, \tag{7}$$

In anticipation of oscillation, we write

$$P^{(1)} = \gamma p^{(1)} e^{iKt}$$
$$\bar{u}^{(1)} = \bar{q}^{(1)} e^{iKt} \tag{8}$$

where K is the complex frequency,

$$K = \Omega + i\Lambda \tag{9}$$

The perturbed wave equation for the pressure fluctuations can be obtained by combining Eqs. (5, 6, and 8) in the usual manner:

$$\nabla^2 p^{(1)} + K^2 p^{(1)} = \nu g^{(1)} \tag{10}$$

with

$$g^{(1)} = iK(\bar{U} \cdot \nabla p^{(1)}) - \nabla \cdot (\bar{U} \cdot \nabla \bar{q}^{(1)} + \bar{q}^{(1)} \cdot \nabla \bar{U}) \tag{11}$$

The boundary condition for Eq. (10) is found by taking $\hat{n} \cdot \nabla P^{(1)}$ from Eq. (6) and combining it with Eq. (3):

$$\hat{n} \cdot \nabla p^{(1)} = -\nu h^{(1)} \quad \text{at } z = 0 \tag{12}$$

with

$$h^{(1)} = -iK\phi A p^{(1)} + \hat{n} \cdot [\bar{U} \cdot \nabla \bar{q}^{(1)} + \bar{q}^{(1)} \cdot \nabla \bar{U}] \tag{13}$$

This type of boundary-value problem can be solved most conveniently by use of Green's function, as has been done in Ref. 1. We find

$$p^{(1)} = \nu \int_v G^*(\bar{r}|\bar{r}_0) g_0^{(1)} dV_0 + \nu \int_s G^*(\bar{r}|\bar{r}_{0_s}) h_0^{(1)} dS_0 \tag{14}$$

where the Green's function $G(\bar{r}|\bar{r}_0) = \sum_\alpha A_\alpha \Psi_\alpha(\bar{r})$ satisfies

$$(\nabla^2 + K^2) G(\bar{r}|\bar{r}_0) = \delta(\bar{r} - \bar{r}_0) \tag{15}$$

$$\hat{n} \cdot \nabla G(\bar{r}|\bar{r}_0) = 0 \quad \text{on the bounding surface} \tag{16}$$

where $\delta(\bar{r} - \bar{r}_0)$ is the Dirac delta function. The eigenfunctions are

$$\Psi_\alpha(\bar{r}) = \cos(k_l z)\cos(m\theta)J_m(k_{mn}r) \tag{17}$$

and the eigenvalues are given by

$$k_l = l\pi R/L, \quad l = 0,1,2\ldots \tag{18a}$$

$$[dJ_m(k_{mn}r)/dr]|_{r=1} = 0 \tag{18b}$$

and

$$K_\alpha^2 = k_l^2 + k_{mn}^2$$

By consideration of orthogonality of the eigenfunctions, one finds the coefficients of the Green's function to be

$$A_\alpha = [\Psi_\alpha^*(\bar{r}_0)/E_\alpha^2(K^2 - K_\alpha^2)] \tag{19}$$

thus

$$G(\bar{r}|\bar{r}_0) = \sum [\Psi_\alpha(\bar{r})\Psi_\alpha^*(\bar{r}_0)/E_\alpha^2(K^2 - K_\alpha^2)] \tag{20}$$

where E_α^2 is the normalization constant

$$E_\alpha^2 = \int_v \Psi_\alpha^* \Psi_\alpha dV \tag{21}$$

The mean flow Mach number in the combustion chamber is on the order of 10^{-2}. If it is assumed that the jet Mach number, which is represented by ν, is of the same order as chamber flow Mach number, then it is clear that

$$p^{(1)} = \Psi_\alpha + 0(\nu) \tag{22}$$

This is borne out if we consider the zeroth order in ν for Eqs. (10) and (12).

An explicit solution can be determined for the complex frequency K by Eq. (22). This is done by substitution of Eq. (20) into Eq. (14) and comparing with Eq. (22); we find for a particular mode, say $\alpha = N$,

$$K^2 = K_N^2 + (\nu/E_N^2)\left[\int_v g^{(1)}\Psi_N^* dV + \int_s h^{(1)}\Psi_N^* dS\right] \tag{23}$$

The stability criterion for linear waves considered in the present analysis is expressed by Eq. (9). Obviously then since $P^{(1)} \sim e^{-\Lambda t}$, the sign of Λ will determine whether the wave is amplified or attenuated. To find the value of K, we first expand in a power series in the second small parameter ν (mean flow Mach number):

$$K = \Omega^{(0)} + \nu\Omega^{(1)} + i[\Lambda^{(0)} + \nu\Lambda^{(1)}] + 0(\nu^2) \tag{24}$$

Then expanding Eq. (23) in Taylor series and comparing with Eq. (24), we find

$$\Lambda^{(0)} = 0, \Omega^{(0)} = K_N$$

and

$$\Lambda^{(1)} = \nu/2K_N E_N^2 \, \mathscr{I} \left[\int_v g^{(1)}\Psi_N^* \, dV \times \int_s h^{(1)} \Psi_N^* dS\right] \tag{25}$$

and so on to higher order in ν where $\mathscr{I}$ refers to the imaginary part.

Evaluating Eq. (25), we find

$$\Omega^{(0)} = k_{mn}$$

$$\Lambda^{(1)} = \frac{\nu}{E_N^2}\left[\int_s \frac{P^{(1)}P^{(1)*}}{2} \bar{U}\cdot \hat{n} dS - A^{(r)} \int_s \frac{P^{(1)}P^{(1)*}}{2} \phi dS\right] \tag{26}$$

where $A^{(r)}$ is the real part of the admittance function A. Equation (26) can also be obtained from the analysis of Cantrell and Hart.[4,5] The wave is, from Eqs. (22) and (26), for standing waves,

$$P^{(1)} = \gamma \cos(k_l z) \cos m(\theta - \theta_0) J_m(k_{mn}r) e^{(-\nu\Lambda^{(1)} + iK_N)t} + 0(\nu) \tag{27}$$

where θ_0 is the angular position of the wave relative to a reference slot of the slot array.

If the injection velocity from the slots is assumed to be uniform, then on the chamber foreward end boundary, we have

$$\nu\bar{U} = \nu\phi\hat{e}_z, \quad \text{at } z = 0 \tag{28}$$

where $\nu\phi$ represents the flow emerging into the chamber from the upstream slots [see Eq. (4)].

Substitution of the time independent part of Eqs. (27) and (28) into Eq. (26) then gives the growth rate for transverse waves generated by the head-end jets. For standing transverse waves,

$$\Lambda^{(1)} = \Lambda_1 + \Lambda_2 \tag{29}$$

with

$$\Lambda_1 = -\frac{\nu[1 + A^{(r)}]}{(2\pi L/R)\ [1 + (\sin 2\pi l/2\pi l)]} N\theta_s \tag{30}$$

$$\Lambda_2 = -\frac{\nu[1 + A^{(r)}]}{\dfrac{2\pi L}{R}\ [1 + (\sin 2\pi l/2\pi l)]} \frac{N}{m} \sin(m\theta_s) \cos 2m\theta_0 \tag{31}$$

In arriving at Eq. (29) a condition on η naturally arises so that selective driving may be predicted. It is apparent that Λ_2 is proportional to

$$\int_0^{2\pi} \cos(\eta N\theta)\cos(2m\theta)d\theta$$

so that for a nonzero result for Λ_2 the condition is that

$$\eta = 2m/N \tag{32}$$

where N is the number of slots.

For a transverse traveling wave, the pressure oscillation is

$$P^{(1)} = \gamma \cos(k_l z) J_m(k_{mn}r) e^{\,im\theta} e^{(-\nu\Lambda^{(1)} + iK_N)t} \tag{33}$$

Substituting Eqs. (28) and (33) into Eq. (26), we find for traveling waves

$$\Lambda^{(1)} = 2\Lambda_1 \tag{34}$$

where Λ_1 is defined in Eq. (30).

Equations (29) (34) then describe the rate of growth of waves in the cylindrical chamber due only to the effects of discrete mass injection from the foreward slot boundary.

For standing transverse acoustic waves, the growth rate $\Lambda^{(1)}$ consists of two components. The first term Λ_1 reflects a mean driving effect that acts on all transverse modes. The magnitude of Λ_1 is directly proportional to $(N\theta_s)$ which represents the total slot area. Thus, a higher value of $N\theta_s$ means a higher rate of mass injection through the slots and, consequently, a higher rate of energy convection into the chamber. This is somewhat analogous to an increase in propellant surface area in ordinary pressure coupled oscillations. Also the magnitude of Λ_1 is negative, which means that it is always a driving effect. The second term Λ_2 accounts for the effect of the discreteness of mass injection on the transverse waves in that only specific standing wave modes are affected, depending on the number of slots present. Here the magnitude of Λ_2 is not always negative so that waves can be driven as well as be attenuated. The relative magnitudes of these two terms are shown in Fig. 2 for $\theta_0 = 0$. The mean driving effect is compared with the selective driving effect for the first five selectively driven wave modes as determined by Eq. (29). It is seen that the effects are of comparable magnitude at small values of $N\theta_s$ but the selective driving effect diminishes rapidly as the number of slots and the width of each slot increases. The magnitude of the term Λ_2 will eventually decrease and change sign as the product $N\theta_s$ continues to increase. The change in sign means that Λ_2 becomes a damping effect. The net effect will always be driving since $|\Lambda_1| \geq |\Lambda_2|$ and Λ_1 is always negative.

It is interesting to note from Eq. (34) that for a traveling transverse wave the selective driving effect is completely absent. Every traveling transverse wave mode will be indiscriminately driven at essentially the same rate.‡ This result is physically plausible since as the wave trains move over the slots each slot will act on all portions of the wave, so the net effect of the selectiveness will be zero. We must mention here, however, that this result is true only if the flow emerging from the slots is normal to the chamber foreward boundary. Otherwise, Eq. (26) does not hold for a traveling transverse wave. Note that traveling waves will always receive more driving from the slot effect than standing waves.

Equation (32) is the relation determining the transverse wave modes that will be driven by N number of slots. It is apparent from this equation that the number of slots, either even or odd, has no bearing on whether a certain wave will be driven or not. But the number of slots will affect the rate at which a certain wave mode will be driven. For example, both a six-slot and a three-slot configuration will drive the third transverse wave mode (0,3,0), but the growth rate due to selective driving effect for the six-slot would be twice the magnitude as that for the three-slot configuration since the effective driving surface is doubled. This can be seen by solving Eq. (31) for both cases for $\theta_0 = 0, l = 0$.

For three-slots,

$$\Lambda_2 = -(\nu R/2\pi L)(1 + A^{(r)})(\tfrac{1}{2}\sin 3\theta_s)$$

For six-slots,

$$\Lambda_2 = -(\nu R/2\pi L)(1 + A^{(r)})(\sin 3\theta_s)$$

With regard to the influence of the combustion chamber geometry on the growth rate, we note that $\Lambda^{(1)}$ is directly proportional to R/L for either type of mode. Then one way to minimize forward end slot driving effects is to design slender combustion chambers so that R/L is small.

It has been aforementioned that the combustion chamber side-wall is a primary source of energy in the amplification of transverse wave modes. It would be useful then to compare the relative magnitudes of the driving effects of the forward end slots to that of the sidewalls.

The expression for the growth rate $\Lambda_w^{(1)}$ due to wall driving has been derived by Culick[1] for traveling (or standing) transverse waves as

$$\Lambda_w^{(1)} = (\nu_w/1 - m_1^2)(m_1^2 + A_w^{(r)}) \tag{35}$$

where ν_w is the Mach number of the combustion gases at the wall surface, $A_w^{(r)}$ is the real part of wall acoustic admittance function, and $m_1 = m/k_{mn}$.

The ratio of the growth rates is largest when a traveling wave is considered, thus taking the ratio between Eqs. (34) and (35),

$$\frac{\Lambda^{(1)}}{\Lambda_w^{(1)}} = \frac{\nu_s}{\nu_w}\frac{R(1 - m_1^2)}{2L}\frac{1 + A_s^{(r)}}{m_1^2 + A_w^{(r)}}\,2N\theta_s \tag{36}$$

For transverse waves of the form $(0,m,0)$ it is reasonable to assume that

$$\{1 + A_s^{(r)}/2[m_1^2 + A_w^{(r)}]\} \sim 1$$

Although precise values of the admittance functions are not known, we expect that $A_w^{(r)}$ is of the same order as $A_s^{(r)}$ or smaller. The term $R(1 - m_1^2)/L$ is of order 10^{-1} to 1. The term $N\theta_s$ can be expected to be again of order 10^{-1}. Finally the term ν_s/ν_w is of order 1 to 10. Hence, we see that from Eq. (36), $\Lambda^{(1)}/\Lambda_w^{(1)}$ can have values on the order of 10^{-2} to 1. For extreme cases then, the foreward end slots can contribute as much as the side wall in the amplification of transverse acoustic waves.

‡ Note that in an actual situation, however, the damping effects may be quite different for each traveling tangential mode. Only those modes for which driving effects exceed damping will in fact be amplified.

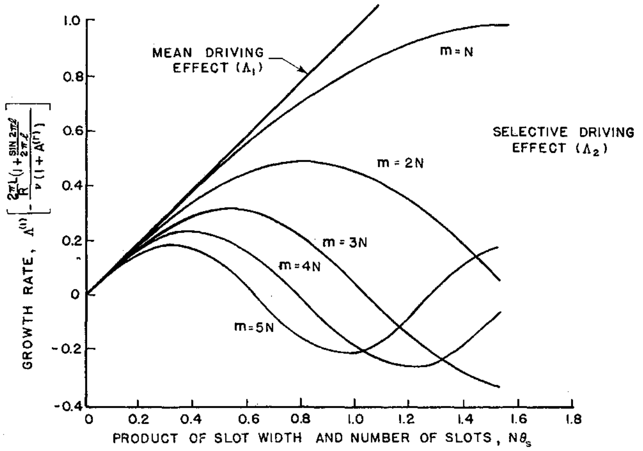

Fig. 2 Comparison between mean and selective driving effects.

Conclusion

In conclusion, we have evaluated a source contributing to solid rocket combustion instability which has been previously overlooked, that of discrete mass injection from the chamber foreward end slots. It is shown that transverse acoustic waves are inevitably driven by mass injection but that certain modes are amplified at a higher rate than others depending on the number and width of the slots. Also, with all other losses and gains remaining unchanged, the traveling transverse waves are more likely to be driven to instability than corresponding standing waves. Any longitudinal mode may be driven by the same mechanism.

In the design of rocket motor grains of the type analyzed in this paper, the number of slots is probably determined by consideration, such as fast attainment of operating pressure, constant propellant burning surface area, grain structural integrity, etc. It is felt, however, that the equally important design consideration of acoustic stability based on the present analysis must also be taken into account. There does not seem to be any general rule in the selection of the number of slots from the standpoint of acoustic stability except to say that no slots at all is best. It is clear from the analysis that Mach number and mass flux for the slot flow must be minimized to minimize acoustic driving from this source.

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